Quantum field theory on curved Spacetime

YOUSSEF Ahmed

Director : J. Iliopoulos

LPTENS Paris

Laboratoire de physique théorique de l'école normale supérieure

Why QFT on curved spacetime

- In its usual formulation QFT simply ignores gravity
- No one knows how to write a full quantum gravity theory
- But we expect the existence a semi-classical regime where one can only quantize matter fields and keep gravity classical (cf a theory of quantum matter interacting with a classical electromagnetic field)
- ψ_{matter} coupled to classical $g_{\mu\nu} \iff \psi_{\text{matter}}$ lives on curved spacetime

Consequences of the coupling to an external field

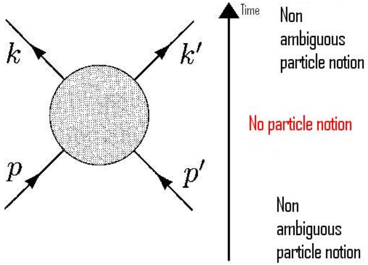
- The coupling to an external field furnish energy that can create particles : Schwinger effet in QED coupled to an external $\vec{E}$ field

$$\mathcal{P}_{e^+e^- \text{pair creation}} \propto \exp\left(-\frac{m^2}{eE}\right)$$

- More importantly, the notion of particle is ambiguous. Remember that origin of the particle concept in QFT is an asymptotic one
  - Free QFT $\longrightarrow \mathcal{E}$ = space of stationnary solutions
  - $\mathcal{E}$ has a Fock space structure
    $\Longrightarrow$ particle interpretation of theory

Consequences of the coupling to an external field

- The same reasoning doesn't hold in an interacting theory



- Example of QCD :
  - Free theory $\longrightarrow$ Quarks
  - Interacting theory $\longrightarrow$ hadrons
- Particle notion defined asymptotically in a free theory

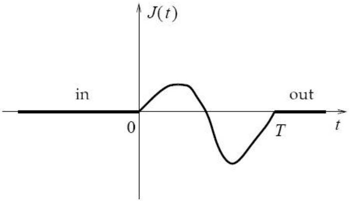
  Particle notion defined asymptotically in a flat spacetime

Quantum driven harmonic oscillator

- equation of motion $\ddot{q} + \omega^2 q = J(t)$

$$\begin{cases} \dot{q} = & p \\ \dot{p} = & -\omega^2 q + J(t) \end{cases}$$

$$a^{\pm} = \sqrt{\frac{\omega}{2}} \left[q(t) \mp \frac{i}{\omega} p(t) \right] \quad \text{and} \quad a^{-}(t=0) = a^{-}_{\text{in}}$$



- Solution

$$a^{-}(t) = a^{-}_{\text{in}} \mathrm{e}^{-i\omega t} + \frac{i}{\sqrt{2\omega}} \int_0^t d\tau J(\tau) \mathrm{e}^{i\omega(\tau - t)}$$

Quantum driven harmonic oscillator

- 2 asymptotic regions

$$a^-(t) = \begin{cases} a^-_{\text{in}} \mathrm{e}^{-i\omega t} & \text{if } t \leq 0 \\ a^-_{\text{out}} \mathrm{e}^{-i\omega t} & \text{if } t \geq T \end{cases}$$

$$a^-_{\text{out}} = a^-_{\text{in}} + J_0 \qquad J_0 = \frac{i}{\sqrt{2\omega}} \int_0^T d\tau J(\tau) \mathrm{e}^{i\omega\tau}$$

- 2 vacuum $|0_{\text{in}}\rangle$ and $|0_{\text{out}}\rangle$

$$a^-_{\text{in}} |0_{\text{in}}\rangle = 0 \qquad \begin{aligned} a^-_{\text{out}} |0_{\text{out}}\rangle &= 0 \\ a^-_{\text{out}} |0_{\text{in}}\rangle &= J_0 |0_{\text{in}}\rangle \end{aligned}$$

$$|0_{\text{in}}\rangle \neq |0_{\text{out}}\rangle$$

- Particle (exitation) creation $N(t) = a^+(t)a^-(t)$

$$\langle 0_{\text{in}}| N(t) |0_{\text{in}}\rangle = \begin{cases} 0 & \text{if } t \leq 0 \\ |J_0|^2 & \text{if } t \geq T \end{cases}$$

Real scalar field

- Minimal coupling

$$S = \int d^4x \sqrt{-g} \left\{ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right\}$$

$$\left(\Box + m^2\right) \phi = 0 \qquad \Box\phi = \frac{1}{\sqrt{-g}} \partial_\mu \left(g^{\mu\nu} \sqrt{-g} \partial_\nu \phi \right)$$

- Mode decomposition
  - Minkowski : Poincaré invariance gives a privileged coordinate system (t, x, y, z)

$$\begin{cases} \phi(t, \vec{x}) & = \sum_{\vec{k}} a_{\vec{k}}\, u_{\vec{k}}(t, \vec{x}) + a^\dagger_{\vec{k}}\, u^*_{\vec{k}}(t, \vec{x}) \\ u_{\vec{k}} & \propto \mathrm{e}^{-i\vec{k}.\vec{x}} \mathrm{e}^{-i\omega t} \end{cases}$$

  - Curved spacetime : many different mode decomposition

$$\phi(x) = \sum_i a_i u_i(x) + a_i^\dagger u_i^*(x) = \sum_i \bar{a}_i \bar{u}_i(x) + \bar{a}_i^\dagger \bar{u}_i^*(x)$$

Bogoliubov transformation

- 2 different vacuum

$$\begin{cases} a_i \left|0\right\rangle & = 0 \quad \forall i \\ \bar{a}_i \left|\bar{0}\right\rangle & = 0 \quad \forall i \end{cases}$$

- $\{u_i\}$ and $\{\bar{u}_i\}$ complete bases of states

$$\bar{u}_j = \sum_i \alpha_{ji} u_i + \beta_{ji} u_i^*$$

$$\Longrightarrow \begin{cases} a_i & = \sum_j \alpha_{ji} \bar{a}_j + \beta_{ji}^* \bar{a}_j^\dagger \\ \bar{a}_j & = \sum_i \alpha_{ji}^* a_i - \beta_{ji}^* a_j^\dagger \end{cases}$$

- $a_i \left|\bar{0}\right\rangle = \sum_j \beta_{ji}^* \left|\bar{1}_j\right\rangle \neq 0$ if a $\beta_{ji} \neq 0$
- Created particle number

$$N_i = a_i^\dagger a_i \qquad \left\langle\bar{0}\right| N_i \left|\bar{0}\right\rangle = \sum_j \left|\beta_{ji}\right|^2$$

Particle creation in spacially flat FRW

- Conformally flat spacetime

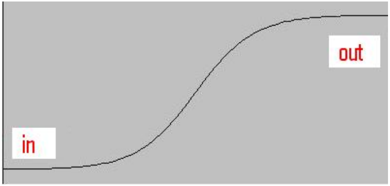
$$ds^2 = dt^2 - a^2(t)dx^2 \qquad d\eta = \frac{dt}{a(t)} \qquad ds^2 = a^2(\eta)\left[d\eta^2 - dx^2\right]$$

$$g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu} \qquad \sqrt{-g} = a^4(\eta)$$

- field equation

$$u_k(\eta, \vec{x}) = \frac{1}{\sqrt{2\pi}} e^{i\vec{k}.\vec{x}} \chi_k(\eta) \qquad \ddot{\chi}_k + \left[k^2 + a^2(\eta)m^2\right]\chi_k = 0$$

- Exact soution in terms of hypergeometric functions if $a^2(\eta) = A + B\tanh(\rho\,\eta)$



Particle creation in spacially flat FRW

- We impose Minkowskian modes as $\eta \to \pm\infty$

$$\begin{cases} u_k^{\text{in}} & = \cdots \longrightarrow \frac{1}{\sqrt{4\pi\omega_{\text{in}}}} \mathrm{e}^{i(kx-\omega_{\text{in}}\eta)} & \text{as} \quad \eta \to -\infty \\ u_k^{\text{out}} & = \cdots \longrightarrow \frac{1}{\sqrt{4\pi\omega_{\text{out}}}} \mathrm{e}^{i(kx-\omega_{\text{out}}\eta)} & \text{as} \quad \eta \to \infty \end{cases}$$

- One can then find analytically α_k and β_k such that

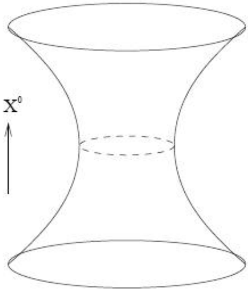
$$u_k^{\text{in}} = \alpha_k u_k^{\text{out}} + \beta_k u_{-k}^{\text{out}*}$$

- Number of created particles

$$N = \sum_k |\beta_k|^2$$

Classical de Sitter spacetime

- de Sitter spacetime : maximally symmetric spacetime with isotropic and homogeneous spacial sections, positive scalar curvature
  Example : Flat spatial sections
  $$ds^2 = -dt^2 + \mathrm{e}^{2Ht} d\vec{x}^2$$
- dS_d may be realized in $\mathcal{M}^{d,1}$ as the hyperboloid
  $$-X_0^2 + X_1^2 + \cdots + X_d^2 = l^2$$
  Here the $O(d,1)$ symmetry is manifest



2 POINT FUNCTION

- Free field theory, so all the information is in the 2 point function. For instance the Wightman function

$$G(X,Y) = \langle 0|\, \phi(X)\phi(Y)\, |0\rangle \qquad \left(\Delta_{\mathrm{dS}_d} - m^2\right) G = 0$$

- $G(X,Y) = G\left(P(X,Y)\right)$ where $P(X,Y)$ is the de Sitter invariant length. Hypergeometric equation :

$$z(1-z)\ddot{G} + \left(\frac{d}{2} - zd\right)\dot{G} - m^2 G = 0 \quad \text{with} \quad z = \frac{1+P}{2}$$

- Solutions : a one parameter family of de Sitter invariant Green functions G_α corresponding to a one parameter family of de Sitter invariant vacuum states $|\alpha\rangle$

$$G_\alpha(X,Y) = \langle \alpha|\, \phi(X)\phi(Y)\, |\alpha\rangle$$

Thermal radiation

- A geodesic observer $x(\tau)$ equipped with a detector of Hamiltonian and energy eigenstates

$$H\,|E_j\rangle = E_j\,|E_j\rangle$$

- The geodesic observer measures a thermal bath of particles when the field ϕ is in the vacuum state $|0\rangle$: the field-detector coupling induces a thermally populated energy levels

$$N_i \propto \mathrm{e}^{-\beta E_i}$$

- The de Sitter temperature is

$$T = \frac{1}{2\pi l}$$

Final notes

- One can compute the vectorial and spinorial propagator too
- Instead of a matter field we can consider linearized gravity itself : de Sitter quantum gravity

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$

  - Perturbative quantum gravity is non renormalizable. This is a short distance property that is independent of the large scale shape of spacetime
  - But one can still treat it as an effective field theory and get the first quantum corrections
  - The graviton $h_{\mu\nu}$ propagator on de Sitter has an infrared pathology even at the tree level

Final notes

- One can try to consider the back reaction of quantum fields (matter and gravitons) on spacetime

$$G_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle$$

- Cosmological constant problem : gravity couples to any form of energy. So a naive renormalization of the vacuum energy is not possible and

$$\frac{E_0}{V} = \int^{\Lambda_{\text{Planck}}} d^3k \frac{1}{2}\sqrt{k^2+m^2} \approx \Lambda^4_{\text{Planck}} \approx 10^{94} \text{g.cm}^3!$$

- The Stone von Neumann theorem breaks down in infinite dimensional context (field theory). Infinitley many inequivalent representations of the quantum algebra exists and no Poincaré invariance to pick one $\longrightarrow$ algebraic approach to QFT on curved space time

Final notes

- Probably profound link between gravity, the quantum and thermodynamics
- dS/CFT correspondence
- Finally one more reason to think that QFT is the quantum theory of fields and not a quantum theory of particles. To mention also : Rovelli's global/local particles in QFT